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THREE-DIMENSIONAL SETS WITH SMALL SUMSET

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We describe the structure of three dimensional sets of lattice points, having a small doubling property. Let $\mathcal{K}$ be a finite subset of $\mathbb{Z}^3$ such that $\dim\mathcal{K}=3$. If $|\mathcal{K}+\mathcal{K}|<\frac{13}{3}|\mathcal{K}|-\frac{25}{3}$ and $|\mathcal{K}|>12^3$, then $\mathcal{K}$ lies on three parallel lines. Moreover, for every three dimensional finite set $\mathcal{K}\subseteq\mathbb{Z}^3$ that lies on three parallel lines, if $|\mathcal{K}+\mathcal{K}|<5|\mathcal{K}|-10$, then $\mathcal{K}$ is contained in three arithmetic progressions with the same common difference, having together no more than $v=|\mathcal{K}+\mathcal{K}|-3|\mathcal{K}|+6$ terms. These best possible results confirm a recent conjecture of Freiman and cannot be sharpened by reducing the quantity v or by increasing the upper bounds for $|\mathcal{K}+\mathcal{K}|$.

1. Introduction and results

The direction of additive number theory where properties of a set are derived from known properties of its sumset is called *Structure Theory of Set Addition*. Freiman's fundamental result on addition of finite sets [3] describes the structure of sets of integers with small doubling property: let A be a set of integers of cardinality $|A|$ such that the sumset $2A=A+A=\{a+a'\,|\,a,a'\in A\}$ has cardinality at most $\alpha|A|$. Then A is contained in a proper d-dimensional arithmetic progression $P=\{a+x_1q_1+\cdots+x_dq_d\,|\,0\le x_i<\ell_i, i=1,2,\ldots,d\}$, with dimension $d\le[\alpha-1]$ and length $\ell(P)=|P|=\ell_1\ell_2\ldots\ell_d\le C(\alpha)|A|$. In other words, A is a large subset of a generalized arithmetic progression. In most applications, it is essential to have a quantitative understanding of the dependence of $C(\alpha)$ on α. Using an earlier approach of Ruzsa [6] and Bilu [1], Chang obtained in [2] the best bound so far known.

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This work deals with the following problem first raised at the talk of Freiman at the Number Theory Conference held at CIRM, Marseille, on June 1993 (see the survey [4, pp. 16] and also [5, pp. 249]). Let $K \subseteq \mathbb{Z}^d$ be a finite d-dimensional set of lattice points. Freiman asked what can be said about the *exact* structure of K, if $|K+K|$ is very close to its minimal possible value? Can we make the result of the main theorem more precise? It is possible to obtain *exact estimates* for the constant $C(\alpha)$?

In this paper we solve the above question for the first open case $d=3$. In a subsequent paper we shall present an extension to the general case $d \geq 3$, which is more complicated. We assume that a finite three-dimensional set K has a doubling coefficient $\frac{|2K|}{|K|}$ less than $\frac{13}{3}$ and we give a sharp estimate for the number of lattice points in the convex hull of K:

Theorem 1.1. *Let $\mathcal{K}$ be a finite subset of $\mathbb{Z}^3$ of affine dimension* $\dim \mathcal{K}=3$.

(a) *If $|\mathcal{K}+\mathcal{K}| < \frac{13}{3}|\mathcal{K}| - \frac{25}{3}$ and $|\mathcal{K}| > 12^3$, then $\mathcal{K}$ lies on three parallel lines.*

(b) *If $\mathcal{K}$ lies on three parallel lines and $|\mathcal{K}+\mathcal{K}| < 5|\mathcal{K}|-10$, then $\mathcal{K}$ is contained in three arithmetic progressions with the same common difference, having together no more than $v=|\mathcal{K}+\mathcal{K}|-3|\mathcal{K}|+6$ terms.*

Let us mention that the cases $d=1$, $\mathcal{K} \subseteq \mathbb{Z}$, $|2\mathcal{K}| < 3|\mathcal{K}|-3$ and $d=2$, $\mathcal{K} \subseteq \mathbb{Z}^2$, $|2\mathcal{K}| < \frac{10}{3}|\mathcal{K}|-5$ were solved earlier by Freiman, see [3], pp. 11 and pp. 28. Corresponding results for $d=2$, $\mathcal{K} \subseteq \mathbb{Z}^2$, $|2\mathcal{K}| < 3.5|\mathcal{K}|-7$ were obtained in [7]; see also [9] and [8].

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2. Definitions and preliminaries

Throughout this paper we use the following notations. $\mathbb{Z}$ denotes the rational integers and $\mathbb{N}$ the nonnegative elements of $\mathbb{Z}$. Let $\mathbb{R}^d$ be the d-dimensional Euclidian space and $\mathbb{Z}^d$ the additive group of integral vectors in $\mathbb{R}^d$. If M is a finite set, the number of its elements will be denoted by $|M|$. We denote by $M+N=\{x+y \mid x \in M, y \in N\}$ the *algebraic sum* of two finite sets M and N. We call $2M=M+M$ the *sum set* of M. For $x \in \mathbb{Z}^d$, we write $M+x$ for the set $\{m+x \mid m \in M\}$. By *arithmetic progression* of k terms in a torsion-free Abelian group G, we understand a set of the form $\{a+tv \mid t=0,1,\dots,k-1\}$ where $a, v \in G$ and $v \neq 0$. The *affine dimension* $\dim A$ of a set $A \subseteq \mathbb{R}^d$ is defined as the dimension of the smallest hyperplane containing A. We denote by $e_0=(0,0,0)$, $e_1=(1,0,0)$, $e_2=(0,1,0)$, $e_3=(0,0,1)$ the vertices of the standard simplex in $\mathbb{R}^3$. A vector v will also be called a *point* and will be written in the form $v=(x,y,z)$ where x,y,z are the coordinates of the vector with respect to the canonical basis $\{e_1,e_2,e_3\}$.

Let us recall the concept *projection* in the sense of Freiman [3]. Let $\{f_1,f_2,f_3\}$ be a basis of $\mathbb{R}^3$ and assume that $\mathcal{K}\subseteq\mathbb{R}^3$ is a finite set. Suppose that for $x\in\mathbb{R}$ and $y\in\mathbb{R}$ there are $s\geq 1$ points of $\mathcal{K}$ having the first two coordinates equal to x and y respectively. Instead of these s points, we choose the points $xf_1+yf_2+jf_3$, with $0\leq j\leq s-1$. This process is performed for all fixed (x,y) with $s\geq 1$. The set $\mathcal{K}_0$ so obtained is called the *projection* of the set $\mathcal{K}$ onto the plane $\mathcal{L}=\{af_1+bf_2\,|\,a\in\mathbb{R},b\in\mathbb{R}\}$ parallel to the vector f_3. It is obvious that $|\mathcal{K}_0|=|\mathcal{K}|$ and Theorem 1.16 from [3] states that $|\mathcal{K}_0+\mathcal{K}_0|\leq|\mathcal{K}+\mathcal{K}|$. Actually, in the original form, Theorem 1.16 is only stated for integral vectors, but it is obviously true in the more general setting of this paragraph.

We complete Section 2 by showing that Theorem 1.1 cannot be sharpened by increasing the upper bounds for $|2\mathcal{K}|$ or by reducing the quantity v.

Example 1. Let $x\geq 3$ be an integer. Define $\mathcal{A}=\mathcal{P}_0\cup\mathcal{P}_1\cup\mathcal{P}_2\cup\{e_3\}$, where $\mathcal{P}_0=\{e_0,e_1,2e_1,\dots,(x-1)e_1\}$, $\mathcal{P}_1=\mathcal{P}_0+e_2$, $\mathcal{P}_2=\mathcal{P}_0+2e_2$. Then $|2\mathcal{A}|=|2\mathcal{P}_0+\{e_0,e_2,2e_2,3e_2,4e_2\}|+|(\mathcal{P}_0\cup\mathcal{P}_1\cup\mathcal{P}_2)+e_3|+|\{2e_3\}|=5(2x-1)+3x+1=13x-4$. We get that $|2\mathcal{A}|=\frac{13}{3}|\mathcal{A}|-\frac{25}{3}$ and it is obvious that $\mathcal{A}$ cannot be covered by three parallel lines. This proves that Theorem 1.1 cannot be improved by increasing the upper bound (a) for $|2\mathcal{K}|$. ∎

Example 2. Let us take $\mathcal{B}=\{e_0,e_3,2e_3,\dots,(b-4)e_3,xe_3\}\cup\{e_1\}\cup\{e_2\}$, with $b=|\mathcal{B}|\geq 5$ and $x>2b-8$. Clearly, $\mathcal{B}$ is a three-dimensional subset of $\mathbb{R}^3$ that lies on three parallel lines and $|2\mathcal{B}|=5|\mathcal{B}|-10$. Nevertheless, we cannot give an upper bound, depending only on $|\mathcal{B}|$ and $|2\mathcal{B}|$, for the number of terms of three arithmetic progressions of the same difference that cover $\mathcal{B}$. Indeed, any such three arithmetic progressions have at least $x+3$ terms and we can choose x arbitrarily large. This means that the conclusion of Theorem 1.1(b) is false, if the condition $|2\mathcal{K}|<5|\mathcal{K}|-10$ is not satisfied. ∎

Example 3. Let us define $\mathcal{C}=\{e_0,e_3,2e_3,\dots,(c-4)e_3,xe_3\}\cup\{e_1\}\cup\{e_2\}$, with $c=|\mathcal{C}|\geq 5$ and $c-3\leq x<2c-7$. $\mathcal{C}$ is a three-dimensional subset of $\mathbb{R}^3$, $\mathcal{C}$ lies on three parallel lines and $|2\mathcal{C}|=3c-3+x<5c-10$. Moreover, any three arithmetic progressions of the same difference that cover $\mathcal{C}$ have at least $x+3=|2\mathcal{C}|-3c+6$ terms. Therefore the estimate for v given in Theorem 1.1(b) is optimal. ∎

3. Some Lemmas

We require the following lemmas. We show first that assertion (a) of Theorem 1.1 is true in a very simple case.

Lemma 3.1. *Let $\mathcal{K}$ be a finite subset of $\mathbb{Z}^3$ of affine dimension* $\dim\mathcal{K}=3$. *Assume that there is a line ℓ such that $\mathcal{K}\setminus\ell$ lies on a plane. If $k=|\mathcal{K}|\geq 25$, then $|2\mathcal{K}|\geq \frac{13}{3}|\mathcal{K}|-\frac{25}{3}$ or $\mathcal{K}$ lies on three parallel lines.*

Proof. *(I).* We first assume that $\mathcal{K}\setminus\ell$ lies on a plane *parallel* to the line ℓ. Let us choose r parallel lines $\ell_1,\ell_2,\dots,\ell_r$ that cover $\mathcal{K}$ with $\ell_1=\ell$. If $r=3$ we have nothing to prove. Assume $r\geq 4$ and denote $\mathcal{K}_i=\mathcal{K}\cap\ell_i$, $k_i=|\mathcal{K}_i|\geq 1$, $\bar{\mathcal{K}}=\mathcal{K}\setminus\mathcal{K}_1$ and $\bar{k}=|\bar{\mathcal{K}}|$. It is clear that $\mathcal{K}_1$ and $\bar{\mathcal{K}}$ lie on different parallel planes, $k=k_1+\bar{k}$ and thus

$$\begin{aligned}|2\mathcal{K}| &= |2\mathcal{K}_1|+|\mathcal{K}_1+\bar{\mathcal{K}}|+|2\bar{\mathcal{K}}| \geq (2k_1-1)+\sum_{i=2}^{r}(k_1+k_i-1)+|2\bar{\mathcal{K}}| \\ &= r(k_1-1)+k+|2\bar{\mathcal{K}}|. \end{aligned} \tag{1}$$

Case (a). If $|2\bar{\mathcal{K}}|\geq\frac{10}{3}\bar{k}-5$, then we get

$$|2\mathcal{K}|\geq(4k_1-4)+k+\left(\frac{10}{3}\bar{k}-5\right)=\left(\frac{13}{3}k-\frac{25}{3}\right)+\frac{2}{3}(k_1-1)\geq\frac{13}{3}k-\frac{25}{3}.$$

Case (b). If $\bar{k}\leq 10$, then using $k\geq 25$ we get

$$|2\mathcal{K}|\geq(4k_1-4)+k+(2\bar{k}-1)=5k-5-2\bar{k}\geq 5k-25\geq\frac{13}{3}k-\frac{25}{3}.$$

Case (c). If $\bar{k}\geq 11$ and $|2\bar{\mathcal{K}}|<\frac{10}{3}\bar{k}-5$, then Theorem 1.17 from [3, pp. 28], implies that $\bar{\mathcal{K}}$ can be covered by two parallel lines γ_1 and γ_2. If $k_1=1$ or if γ_1 and γ_2 are parallel to ℓ, then the set $\mathcal{K}$ itself lies on three parallel lines.

Therefore we may assume that γ_1 and γ_2 are not parallel to ℓ and $k_1\geq 2$. It follows that

$$1\leq k_i\leq 2 \ \text{ for every } \ 2\leq i\leq r,\ r-1\leq\bar{k}=k_2+\cdots+k_r\leq 2r-2. \tag{2}$$

We distinguish two subcases depending on $\dim\bar{\mathcal{K}}$.

(c1). If $\dim\bar{\mathcal{K}}=1$ and $k_1=2$, then $\mathcal{K}$ lies on three parallel lines. If $\dim\bar{\mathcal{K}}=1$ and $k_1\geq 3$, then $|2\mathcal{K}|\geq|2\mathcal{K}_1|+|\mathcal{K}_1+\bar{\mathcal{K}}|+|2\bar{\mathcal{K}}|\geq(2k_1-1)+k_1\bar{k}+(2\bar{k}-1)=(k_1-3)(\bar{k}-3)+5(k_1+\bar{k})-11\geq 5k-11\geq\frac{13}{3}k-\frac{25}{3}$.

(c2). If $\dim\bar{\mathcal{K}}=2$, then $|2\bar{\mathcal{K}}|\geq 3\bar{k}-3$ (see [3, pp. 24], inequality (1.14.1)). Inequality (2) implies $r\geq\frac{\bar{k}}{2}+1$. Using $r\geq 4$, $k_1\geq 2$ and inequality (1) we get

$$\begin{aligned}|2\mathcal{K}| &\geq r(k_1-1)+k+|2\bar{\mathcal{K}}|\geq r(k_1-1)+k+(3\bar{k}-3)\\ &=(r-4)(k_1-2)+r+(4k_1-8)+k+(3\bar{k}-3)\\ &\geq\frac{\bar{k}}{2}+1+(4k_1-8)+k+(3\bar{k}-3)=4.5\bar{k}+5k_1-10>\frac{13}{3}k-\frac{25}{3}.\end{aligned}$$

(II). We finish the proof of Lemma 3.1 by examining the case when the line ℓ intersects the plane $\mathcal{P}$ that contains $\mathcal{K}\setminus\ell$. Without loss of generality, we may assume that

(i) the plane $\mathcal{P}=(z=0)$,
(ii) the set $\bar{\mathcal{K}}=\mathcal{K}\cap\mathcal{P}$ satisfies $\bar{k}=|\bar{\mathcal{K}}|\geq 3$,
(iii) the set $\bar{\bar{\mathcal{K}}}=\mathcal{K}\setminus\mathcal{P}=\{z_1e_3,\dots,z_te_3\}\subseteq\ell$ satisfies $t=|\bar{\bar{\mathcal{K}}}|\geq 2$.

(Indeed, if $\bar{k}\leq 2$, then $\mathcal{K}$ lies on three parallel lines and if $t=1$ then we may use case (I).) We can easily prove that

$$\begin{aligned}|2\mathcal{K}| &\geq |2\bar{\mathcal{K}}|+\sum_{j=1}^{t}|\bar{\mathcal{K}}+z_je_3|+|(2\bar{\bar{\mathcal{K}}})\setminus\{e_0,z_1e_3,\dots,z_te_3\}|\\ &\geq |2\bar{\mathcal{K}}|+t|\bar{\mathcal{K}}|+(2t-1)-(t+1)=|2\bar{\mathcal{K}}|+t\bar{k}+(t-2).\end{aligned}\tag{3}$$

If $t=2$ and $\dim\bar{\mathcal{K}}=1$, then $\mathcal{K}$ lies on three parallel lines. If $t=2$ and $\dim\bar{\mathcal{K}}=2$, then again by [3], inequality (1.14.1), $|2\mathcal{K}|\geq(3\bar{k}-3)+2\bar{k}=5\bar{k}-3=5k-13>\frac{13}{3}k-\frac{25}{3}$. It will be enough to consider only the case $t\geq 3$. If $t\geq 3$ and $\bar{k}\geq 4$, we use (3) and get

$$\begin{aligned}|2\mathcal{K}| &\geq |2\bar{\mathcal{K}}|+t|\bar{\mathcal{K}}|+(t-2)\geq(2\bar{k}-1)+t\bar{k}+(t-2)\\ &=(t-3)(\bar{k}-4)+5(\bar{k}+t)-15\geq 5k-15>\frac{13}{3}k-\frac{25}{3}.\end{aligned}$$

Assume $t\geq 3$ and $\bar{k}=3$. If one of the points of $\bar{\mathcal{K}}$ lies on ℓ, then $\mathcal{K}$ lies on three parallel lines. If $\bar{\mathcal{K}}\cap\ell=\emptyset$, then

$$\begin{aligned}|2\mathcal{K}| &\geq |2\bar{\bar{\mathcal{K}}}|+|\bar{\mathcal{K}}+\bar{\bar{\mathcal{K}}}|+|2\bar{\mathcal{K}}|\geq(2t-1)+3t+(2\bar{k}-1)\\ &=(2t-1)+3t+5=5t+4=5k-11>\frac{13}{3}k-\frac{25}{3}.\end{aligned}$$ ∎

The following result generalize inequalities (3) and (4) from [3, pp. 25].

Lemma 3.2. *Let $\mathcal{K}\subseteq\mathbb{Z}^3$ be a finite set of affine dimension $d=\dim\mathcal{K}=3$. Assume that there are r parallel lines $\ell_1,\dots,\ell_r$ such that $|\mathcal{K}\cap\ell_i|=k_i\geq 1$ for every $1\leq i\leq r$ and $k=|\mathcal{K}|=k_1+\dots+k_r$. If $k_{\max}=\max\{k_1,\dots,k_r\}$, then*

(a) $|\mathcal{K}+\mathcal{K}|\geq\left(5-\frac{2}{r-1}\right)|\mathcal{K}|-(2r+1)+\frac{2}{r-1}$;
(b) $|\mathcal{K}+\mathcal{K}|\geq(3|\mathcal{K}|-3)+(r-2)(k_{\max}-1)$.

Proof. *(a)*. Using inequality (1.14.1), from [3, pp. 24] and $\dim\mathcal{K}=3$ we get $|2\mathcal{K}|\geq 4|\mathcal{K}|-6$.

If $k\leq 2r-1$, this inequality implies

$$|\mathcal{K}+\mathcal{K}|\geq\left(5-\frac{2}{r-1}\right)|\mathcal{K}|-(2r+1)+\frac{2}{r-1}.$$

Therefore, we may assume that $k\geq 2r$ and $k_j\geq 2$, for some $1\leq j\leq r$.

We *claim* that there is a three-dimensional set $\mathcal{T}\subseteq\mathbb{N}^3$ such that:

(i) $|\mathcal{T}|=|\mathcal{K}|$ and $|2\mathcal{T}|\leq|2\mathcal{K}|$.
(ii) $\mathcal{T}$ lies on exactly r lines $\delta_1,\dots,\delta_r$ parallel to e_3.
(iii) The set $\mathcal{T}\cap(\delta_1\cup\dots\cup\delta_{r-1})$ is two-dimensional.

Assertion (a) follows easily from (i)–(iii). Indeed, we split the set $\mathcal{T}$ into two subsets $\mathcal{T}'=\mathcal{T}\cap(\delta_1\cup\dots\cup\delta_{r-1})$ and $\mathcal{T}''=\mathcal{T}\setminus\mathcal{T}'=\mathcal{T}\cap\delta_r$. The set $\mathcal{T}'$ lies on a plane on exactly $r-1$ parallel lines; we use inequality (1.15.4) from [3, pp. 25], and obtain $|2\mathcal{T}'|\geq\left(4-\frac{2}{r-1}\right)|\mathcal{T}'|-(2r-3)$. Moreover,

$$|\mathcal{T}'+\mathcal{T}''|=\sum_{i=1}^{r-1}|\mathcal{T}''+(\mathcal{T}\cap\delta_i)|\geq\sum_{i=1}^{r-1}(|\mathcal{T}''|+|\mathcal{T}\cap\delta_i|-1)=|\mathcal{T}'|+(r-1)(|\mathcal{T}''|-1).$$

We conclude that

$$\begin{aligned}|\mathcal{T}'+\mathcal{T}''|+|2\mathcal{T}''|&\geq|\mathcal{T}'|+(r-1)(|\mathcal{T}''|-1)+(2|\mathcal{T}''|-1)\\&=|\mathcal{T}|+r(|\mathcal{T}''|-1)\end{aligned}\tag{4}$$

and thus

$$\begin{aligned}|2\mathcal{K}|\geq|2\mathcal{T}|&=|2\mathcal{T}'|+(|\mathcal{T}'+\mathcal{T}''|+|2\mathcal{T}''|)\\&\geq\left(\left(4-\frac{2}{r-1}\right)|\mathcal{T}'|-(2r-3)\right)+(|\mathcal{T}|+r(|\mathcal{T}''|-1))\\&=\left(\left(5-\frac{2}{r-1}\right)|\mathcal{T}|-(2r+1)+\frac{2}{r-1}\right)+(r^2-5r+6)\frac{|\mathcal{T}''|-1}{r-1}\\&\geq\left(5-\frac{2}{r-1}\right)|\mathcal{K}|-(2r+1)+\frac{2}{r-1}.\end{aligned}$$

Let us check now assertions (i)–(iii). After an affine isomorphism of $\mathbb{R}^3$, we may assume that the lines $\ell_1,\dots,\ell_r$ that cover $\mathcal{K}$ are parallel to the vector e_3. Denote by $\mathcal{L}$ the projection of $\mathcal{K}$ onto the plane $(z=0)$ using a projection parallel to the vector e_3. Note that $\dim\mathcal{K}=3$ implies that $\mathcal{L}\cap(z=0)$ is a two-dimensional set. Moreover, using a suitable affine isomorphism of $\mathbb{R}^3$ we may assume without loss of generality that the set $\mathcal{L}\cap(z=0)$ lies in the first quadrant $\{(x,y,0)\,|\,x\geq 0, y\geq 0\}$ and contains the points e_0, e_1 and e_2. A projection of $\mathcal{L}$ onto the plane $(x=0)$ parallel to e_1 followed by a projection onto the plane $(y=0)$ parallel to e_2, maps $\mathcal{L}$ onto a three dimensional set $\mathcal{M}\subseteq\mathbb{R}^3$ that lies on r lines parallel to e_3. It is clear that for every point (x,y,z) of $\mathcal{M}$ the coordinates x, y and z are non-negative integers, $|\mathcal{M}|=|\mathcal{K}|$ and $|2\mathcal{M}|\leq|2\mathcal{K}|$; thus $\mathcal{M}$ satisfies assertions (i)–(ii) of our claim. We prove now that a finite number of additional projections will map $\mathcal{M}$ onto a set $\mathcal{T}$ that satisfies also part (iii) of our claim. The set $\mathcal{M}\cap(z=0)$ lies in the first quadrant $\{(x,y,0)\,|\,x\geq 0, y\geq 0\}$ and let us cover it by $n\geq 2$ lines parallel to the vector e_1. Choose $w=(a,0,0)\in\mathcal{M}$ such that $\mathcal{M}\cap\{te_1\,|\,t\in\mathbb{R}\}$ lies between the points e_0 and w. Let $\mathcal{M}'$ denote the projection of $\mathcal{M}$ onto the plane $(y=0)$ parallel to e_2-w. It is obvious that the intersection of $\mathcal{M}'$ with the plane $(z=0)$ still contains the points $\{e_0,e_1,e_2\}$ and lies in the first quadrant. Note that if $n\geq 3$, then the new set $\mathcal{M}'$ can be covered by n' planes, $2\leq n'<n$, parallel to the plane $(y=0)$ and if $n=2$, then $\mathcal{M}'$ satisfies assertion (iii). After a finite number of projections of this type we will obtain the desired set $\mathcal{T}$.

(b). The proof of inequality (b) of Lemma 3.2 is similar to the proof of (a). In order to estimate $|2\mathcal{T}'|$ we note that $k_{\max}=\max\{k_1,k_2,\dots,k_r\}=|\mathcal{T}\cap\delta_1|$, where δ_1 denotes the line passing through points e_0 and e_3. We use inequality (1.15.3) from [3, pp. 25]. If t_i denotes $|\mathcal{T}\cap\delta_i|$, then $|2\mathcal{T}'|\geq 2|\mathcal{T}'|+(t_1+t_{r-1})(r-2)-(2r-3)$. Inequality (b) follows using (4) and from

$$\begin{aligned}|2\mathcal{K}| &\geq |2\mathcal{T}| = |2\mathcal{T}'| + \big(|\mathcal{T}'+\mathcal{T}''| + |2\mathcal{T}''|\big)\\ &\geq \big(2|\mathcal{T}'| + (t_1+t_{r-1})(r-2) - (2r-3)\big) + \big(|\mathcal{T}| + r|\mathcal{T}''| - r\big)\\ &= 3|\mathcal{T}| + (r-2)|\mathcal{T}''| + (t_1+t_{r-1})(r-2) - 3r + 3\\ &= (3|\mathcal{T}|-3) + (t_1+t_{r-1}+t_r-3)(r-2)\\ &\geq (3|\mathcal{T}|-3) + (k_{\max}-1)(r-2) = (3k-3) + (k_{\max}-1)(r-2). \qquad \blacksquare\end{aligned}$$

As a direct corollary of Lemma 3.2, we obtain the following:

Lemma 3.3. *Let $\mathcal{K}\subseteq\mathbb{Z}^3$ be a finite set of affine dimension $\dim\mathcal{K}=3$. Assume that $|\mathcal{K}+\mathcal{K}|<\frac{13}{3}|\mathcal{K}|-\frac{25}{3}$. If there is a line ℓ such that $|\mathcal{K}\cap\ell|\geq 5$, then $\mathcal{K}$ lies on three parallel lines.*

Proof. The set $\mathcal{K}$ is decomposable into $r\geq 3$ subsets $\mathcal{K}_1,\dots,\mathcal{K}_r$ which lie on r lines $\ell_1,\dots,\ell_r$ parallel to ℓ. We have $k_{\max}=\max\{|\mathcal{K}\cap\ell_i|:1\leq i\leq r\}\geq 5$.

(a). If $4 \le r \le \frac{k-1}{3}+1$, then assertion (a) of Lemma 3.2 implies that

$$|2\mathcal{K}| \ge \Big(5 - \frac{2}{r-1}\Big)|\mathcal{K}| - (2r+1) + \frac{2}{r-1} \ge \frac{13}{3}|\mathcal{K}| - \frac{25}{3},$$

which contradicts the small doubling property of $\mathcal{K}$.

(b). If $r > \frac{k-1}{3}+1$, then assertion (b) of Lemma 3.2 implies that

$$\begin{aligned}|2\mathcal{K}| &\ge (3k-3) + (r-2)(k_{\max}-1) \ge (3k-3) + 4(r-2)\\ &> (3k-3) + 4\Big(\frac{k-1}{3} - 1\Big) = \frac{13}{3}k - \frac{25}{3},\end{aligned}$$

which again contradicts the small doubling property of $\mathcal{K}$.

It follows that $r=3$ and this completes the proof of Lemma 3.3. ∎

The next result provides the inductive step in the proof of Lemma 3.5.

Lemma 3.4. *Let $\mathcal{K}$ be a finite subset of $\mathbb{Z}^3$ of affine dimension $\dim\mathcal{K}=3$ and having the small doubling property $|\mathcal{K}+\mathcal{K}| < \frac{13}{3}|\mathcal{K}| - \frac{25}{3}$. Then*

(a) $\mathcal{K}$ lies on three parallel lines, or
(b) there exists a proper subset $\mathcal{K}^* \subseteq \mathcal{K}$ such that $\dim\mathcal{K}^*=3$, $|\mathcal{K}^*| \ge |\mathcal{K}|-4$ and $|2\mathcal{K}| \ge |2\mathcal{K}^*| + 4.5(|\mathcal{K}|-|\mathcal{K}^*|)$, or
(c) $|\mathcal{K}| \le 4^3$.

Proof. Let $\mathcal{C}$ be the convex hull of $\mathcal{K}$. Then $\mathcal{C}$ is a polyhedron whose vertices are all contained in $\mathcal{K}$. We will use the following notations: if A is an arbitrary vertex of $\mathcal{C}$ we denote by $E_1^A,\dots,E_d^A$ the edges of $\mathcal{C}$ that meet at A. $\mathcal{C}$ is a three dimensional polyhedron and so the degree of A is $d=\deg(A)\ge 3$. For every i, $1\le i\le d$, choose $A_i \in \mathcal{K}$ on the edge E_i^A such that there is no other point of $\mathcal{K}$ between A and A_i. We will say that $A_1,\dots,A_d$ are the neighbours of A in $\mathcal{C}$.

Case A. If for *some vertex* A of $\mathcal{C}$ we have $\deg(A)\ge 4$, then the removal of A from $\mathcal{K}$ reduces the cardinality of $2\mathcal{K}$ by at least 5. Indeed, if $\mathcal{K}^*=\mathcal{K}\setminus\{A\}$, then

$$\begin{aligned}|2\mathcal{K}| &\ge |2\mathcal{K}^*| + |\{A+A, A+A_1,\dots,A+A_d\}| \ge |2\mathcal{K}^*| + d + 1\\ &\ge |2\mathcal{K}^*| + 5 = |2\mathcal{K}^*| + 5(|\mathcal{K}|-|\mathcal{K}^*|) > |2\mathcal{K}^*| + 4.5(|\mathcal{K}|-|\mathcal{K}^*|).\end{aligned}$$

Case B. We assume that for *every vertex* A of $\mathcal{C}$ we have $\deg(A)=3$. If there is a vertex B of $\mathcal{C}$ such that the set $\mathcal{K}$ does not lie in the lattice λ generated by the points B, B_1, B_2, B_3, then there is $B^* \in \mathcal{K}\setminus\lambda$,

$$B^* = \alpha(B_1 - B) + \beta(B_2 - B) + \gamma(B_3 - B),$$

for which at least one of the three coordinates α, β and γ is not an integer. Moreover, we may assume that we choose $B^* \in \mathcal{K} \setminus \lambda$ in such a way that there is no other vector $B' \neq B$ in $\mathcal{K}\setminus\lambda$, with coordinates α', β', γ' satisfying $\alpha' \leq \alpha$, $\beta' \leq \beta$ and $\gamma' \leq \gamma$. Then $2B$, $B+B_1$, $B+B_2$, $B+B_3$ and $B+B^*$ are in $2\mathcal{K}$, but not in $2(\mathcal{K}\setminus\{B\})$. Therefore, if we put $\mathcal{K}^* = \mathcal{K}\setminus\{B\}$, then

$$|2\mathcal{K}| \geq |2\mathcal{K}^*| + 5 > |2\mathcal{K}^*| + 4.5(|\mathcal{K}| - |\mathcal{K}^*|).$$

Case C. We assume that for *every vertex* A of $\mathcal{C}$ we have $\deg(A) = 3$ and $\mathcal{K}$ lies in the lattice generated by A and his neighbours A_1, A_2, A_3. Choose an edge AB of the convex hull of $\mathcal{K}$. Let $\{A_1, A_2, A_3\}$ and $\{B_1, B_2, B_3\}$ be the neighbours of A and B, respectively. We may assume that

(i) A_3 and B_3 lie on the edge AB;
(ii) the points A, B, A_i, B_i lie on the same face of $\mathcal{C}$, for $i=1$ and $i=2$;
(iii) $A = e_0$ and $A_i - A = e_i$, for $i = 1, 2, 3$;
(iv) $\mathcal{K}$ lies in $\mathbb{Z}^3 \cap \{(x,y,z) \mid x \geq 0, y \geq 0, z \geq 0\}$.

If $\mathcal{M} = P + \{m_1 e_3, \ldots, m_t e_3\}$ is a set lying on a line parallel to e_3, let us define it's *length* by $l(\mathcal{M}) = \max\{m_1, \ldots, m_t\} - \min\{m_1, \ldots, m_t\}$. We denote

$$\begin{aligned} \mathcal{K}_A &= \mathcal{K} \cap \{te_3 \mid t \geq 0\}, & k_0 &= |\mathcal{K}_A| \geq 2, \\ \mathcal{K}_{A_i} &= \mathcal{K} \cap \{A_i + te_3 \mid t \geq 0\}, & k_i &= |\mathcal{K}_{A_i}|,\ i = 1, 2. \end{aligned}$$

We distinguish several subcases, but first we will show that $B_i \in \mathcal{K}_{A_i}$ for $i = 1, 2$. Indeed, if $B_1 \notin \mathcal{K}_{A_1}$ (or if $B_2 \notin \mathcal{K}_{A_2}$), then A_1 (or A_2) does not lie in the lattice generated by $\{B, B_1, B_2, B_3\}$ and this contradicts the assumptions of Case C for the vertex B.

Case C1. Assume $k_1 = k_2 = 1$. This is equivalent to $B_1 = A_1$ and $B_2 = A_2$; it follows that $\mathcal{K}$ lies on three parallel lines $\ell_0 = \{te_3 \mid t \geq 0\}$, $\ell_1 = \{A_1 + te_3 \mid t \geq 0\}$ and $\ell_2 = \{A_2 + te_3 \mid t \geq 0\}$.

Case C2. If $k_0 \geq 5$, then Lemma 3.3 implies that $\mathcal{K}$ lies on 3 parallel lines.

Case C3. We assume that $\max\{k_1, k_2\} \geq 2$, $k_0 \leq 4$ and $\mathcal{K}_A$ is not an arithmetic progression. The removal of $\mathcal{K}_A$ from $\mathcal{K}$ reduces the cardinality of $2\mathcal{K}$ by at least $4.5|\mathcal{K}_A|$. Indeed, $\mathcal{K}_A$ is not an arithmetic progression implies (i) $|2\mathcal{K}_A| \geq 2k_0$ and (ii) $|\mathcal{K}_A + \mathcal{K}_j| \geq k_0 + k_j$, if $k_j \geq 2$, for $j = 1$ or $j = 2$. Denote $\mathcal{K}^* = \mathcal{K}\setminus\mathcal{K}_A$. Using $k_0 \leq 4$ we get

$$\begin{aligned} |2\mathcal{K}| &\geq |2\mathcal{K}^*| + |2\mathcal{K}_A| + |\mathcal{K}_A + (\mathcal{K}_{A_1} \cup \mathcal{K}_{A_2})| \\ &\geq |2\mathcal{K}^*| + 2k_0 + (2k_0 + \max\{k_1, k_2\}) \\ &\geq |2\mathcal{K}^*| + (4k_0 + 2) \geq |2\mathcal{K}^*| + 4.5k_0. \end{aligned}$$

Case C4. We assume that $\max\{k_1,k_2\}\geq 2$, $k_0\leq 4$, $\mathcal{K}_A$ is an arithmetic progression and $\max\{l(\mathcal{K}_{A_1}),l(\mathcal{K}_{A_2})\}>l(\mathcal{K}_A)$. It is obvious that $|\mathcal{K}_A+(\mathcal{K}_{A_1}\cup\mathcal{K}_{A_2})|\geq 3k_0$ and thus the removal of $\mathcal{K}_A$ from $\mathcal{K}$ reduces the cardinality of $2\mathcal{K}$ by at least $4.5|\mathcal{K}_A|$. Indeed, $\mathcal{K}^*=\mathcal{K}\setminus\mathcal{K}_A$ satisfies

$$\begin{aligned}|2\mathcal{K}| &\geq |2\mathcal{K}^*|+|2\mathcal{K}_A|+|\mathcal{K}_A+(\mathcal{K}_{A_1}\cup\mathcal{K}_{A_2})|\\ &\geq |2\mathcal{K}^*|+(2k_0-1)+3k_0=|2\mathcal{K}^*|+(5k_0-1)\geq|2\mathcal{K}^*|+4.5k_0.\end{aligned}$$

Case C5. We assume that $\mathcal{K}_A$ is an arithmetic progression, $k_0\leq 4$ and $\max\{l(\mathcal{K}_{A_1}),l(\mathcal{K}_{A_2})\}\leq l(\mathcal{K}_A)$. In this case, $l(\mathcal{K}_A)\leq 3$ and therefore $\mathcal{K}$ lies between two parallel planes $(z=0)$ and $(z=3)$.

We complete the proof of Lemma 3.4 without difficulty; in cases A, B, C3 and C4 we proved that there is a proper subset $\mathcal{K}^*\subseteq\mathcal{K}$ such that $|\mathcal{K}^*|\geq|\mathcal{K}|-4$ and $|2\mathcal{K}|\geq|2\mathcal{K}^*|+4.5(|\mathcal{K}|-|\mathcal{K}^*|)$. Moreover, note that $\mathcal{K}\setminus\mathcal{K}^*$ is a vertex or lies on an edge of the convex hull of $\mathcal{K}$. If $\dim\mathcal{K}^*=3$, then assertion (b) of Lemma 3.4 is true. If $\dim\mathcal{K}^*\leq 2$, we may use Lemma 3.1: inequality $|2\mathcal{K}|<\frac{13}{3}k-\frac{25}{3}$ implies that $k<25$ or $\mathcal{K}$ lies on three parallel lines. In cases C1 and C2, the set $\mathcal{K}$ lies on three parallel lines. If case C5 is true, then we continue the analysis of cases A, B and C for the edges $\mathcal{K}\cap(x=0)\cap(z=0)$ and $\mathcal{K}\cap(y=0)\cap(z=0)$. If assertion (a) and assertion (b) don't hold, we obtain that $|\mathcal{K}|<25$ or $\mathcal{K}$ lies inside a bounded parallelepiped $0\leq x,y,z\leq 3$ and so $|\mathcal{K}|\leq 4^3$. This completes the proof of Lemma 3.4. ∎

Lemma 3.5. *If $\mathcal{K}$ is a finite three-dimensional subset of $\mathbb{Z}^3$ with small doubling property $|\mathcal{K}+\mathcal{K}|<\frac{13}{3}|\mathcal{K}|-\frac{25}{3}$ and if $|\mathcal{K}|>12^3$, then there exists a line ℓ such that $|\ell\cap\mathcal{K}|\geq\frac{1}{81}|\mathcal{K}|$.*

Proof. Let $\mathcal{K}'$ be a minimal subset of $\mathcal{K}$ satisfying

$$\dim\mathcal{K}'=3,\quad |2\mathcal{K}'|\leq|2\mathcal{K}|-4.5(|\mathcal{K}|-|\mathcal{K}'|)\ \text{ and }\ |\mathcal{K}'|\geq\frac{1}{27}|\mathcal{K}|. \tag{5}$$

$\mathcal{K}'$ exists, because $\mathcal{K}$ itself satisfies conditions (5). Put $k=|\mathcal{K}|$ and $k'=|\mathcal{K}'|$. We will show below that $\mathcal{K}'$ lies on three parallel lines; this implies that there exists a line $\ell\subseteq\mathbb{R}^3$ such that $|\mathcal{K}'\cap\ell|\geq\frac{1}{3}|\mathcal{K}'|$ and thus

$$|\mathcal{K}\cap\ell|\geq|\mathcal{K}'\cap\ell|\geq\frac{1}{3}|\mathcal{K}'|\geq\frac{1}{3}\frac{1}{27}|\mathcal{K}|=\frac{1}{81}|\mathcal{K}|.$$

This concludes the proof of Lemma 3.5. It remains to show that $\mathcal{K}'$ lies on three parallel lines. Note that $\mathcal{K}'$ satisfies the small doubling property

$$|2\mathcal{K}'|\leq|2\mathcal{K}|-4.5(k-k')<\left(\frac{13}{3}k-\frac{25}{3}\right)-4.5(k-k')\leq\frac{13}{3}k'-\frac{25}{3}.$$

Moreover, $|\mathcal{K}'|\geq\frac{1}{27}k>\frac{1}{27}12^3=4^3$ and $\dim\mathcal{K}'=3$. By Lemma 3.4, we get that

(a) $\mathcal{K}'$ on three parallel lines, or
(b) there exists a *proper* subset $\mathcal{K}'' \subseteq \mathcal{K}'$ such that $\dim \mathcal{K}'' = 3$, $k'' = |\mathcal{K}''| \geq k' - 4$ and $|2\mathcal{K}''| \leq |2\mathcal{K}'| - 4.5(k' - k'')$.

We complete the proof by showing that assertion (b) is false. Indeed, if we assume that (b) is true, then

$$|2\mathcal{K}''| \leq |2\mathcal{K}'| - 4.5(k' - k'') \leq \big(|2\mathcal{K}| - 4.5(k - k')\big) - 4.5(k' - k'') \\ = |2\mathcal{K}| - 4.5(k - k'').$$

Therefore we should have $k'' < \frac{1}{27}k$, otherwise there would be a contradiction to the minimal choice of $\mathcal{K}'$. Note that $\frac{1}{27}k \leq k' \leq k'' + 4 < \frac{1}{27}k + 4$. We obtain $|2\mathcal{K}'| \leq |2\mathcal{K}| - 4.5(k - k') < \left(\frac{13}{3}k - \frac{25}{3}\right) - 4.5k + 4.5k' < \left(\frac{13}{3}k - \frac{25}{3}\right) - 4.5k + 4.5\left(\frac{k}{27} + 4\right) = \frac{29}{3}$. Using once again inequality (1.14.1), from [3, pp. 24], we get the estimate $|2\mathcal{K}'| \geq 4k' - 6 \geq 4\frac{k}{27} - 6$ and obtain $k < \frac{423}{4}$. This contradicts the assumption $k > 12^3$ and completes the proof of Lemma 3.5. ∎

4. Proof of Theorem 1.1

Proof of (a). Assume that $k = |\mathcal{K}| > 12^3$. In view of Lemma 3.5, we may choose a line $\ell \subseteq \mathbb{R}^3$ such that $|\ell \cap \mathcal{K}| \geq \frac{1}{81}k$. Let us cover the set $\mathcal{K}$ by r lines $\ell_1, \ldots, \ell_r$ parallel to ℓ. If $r \leq 3$, we have nothing to prove. It remains to show that $r \geq 4$ contradicts our hypothesis concerning the sumset of $\mathcal{K}$. It is clear that $k_{\max} = \max\{|\ell_1 \cap \mathcal{K}|, \ldots, |\ell_r \cap \mathcal{K}|\} \geq \frac{1}{81}k$. Put $\delta = \frac{1}{81}$. Using Lemma 3.2 part (b), we get $|2\mathcal{K}| \geq (3k-3) + (r-2)(k_{\max} - 1) \geq (3k-3) + (r-2)(\delta k - 1)$. Note that $k > 12^3 > \frac{2}{\delta}$ and therefore $\delta k - 1 > \frac{1}{2}\delta k > 0$. Using the small doubling property $|2\mathcal{K}| < \frac{13}{3}k - \frac{25}{3}$, we obtain $\frac{13}{3}k - \frac{25}{3} > |2\mathcal{K}| \geq (3k-3) + (r-2)(\delta k - 1)$ and thus

$$r \leq 2 + \frac{|2\mathcal{K}| - (3k-3)}{\delta k - 1} < 2 + \frac{\frac{4}{3}k - \frac{16}{3}}{\delta k - 1} < 2 + \frac{\frac{4}{3}k}{\frac{1}{2}\delta k} = 2 + \frac{8}{3\delta} < 1 + \frac{k-1}{3}.$$

We obtained that $4 \leq r \leq 1 + \frac{k-1}{3}$. In view of Lemma 3.2(a), we get $|2\mathcal{K}| \geq \left(5 - \frac{2}{r-1}\right)|\mathcal{K}| - (2r+1) + \frac{2}{r-1} \geq \frac{13}{3}k - \frac{25}{3}$, which contradicts the small doubling property of $\mathcal{K}$. This completes the proof of Theorem 1.1(a). ∎

Proof of (b). After a coordinate change, we may assume without loss of generality that

(i) $\mathcal{K}$ lies on three lines ℓ_0, ℓ_1, ℓ_2 parallel to e_3 and $e_i \in \mathcal{K}_i = \ell_i \cap \mathcal{K}$, $0 \leq i \leq 2$;
(ii) for every $0 \leq i \leq 2$, there exists a set K_i of nonnegative integers such that $\mathcal{K}_i = \{e_i + ze_3 \mid z \in K_i\}$;

(iii) the greatest common divisor of $K_1 \cup K_2 \cup K_3$ is equal to one.

If $l(K_i) = \max(K_i) - \min(K_i)$ denotes the length of K_i, it is enough to prove that the small doubling property $|2\mathcal{K}| < 5|\mathcal{K}| - 10$ implies

$$l(K_0) + l(K_1) + l(K_2) \leq |2\mathcal{K}| - 3|\mathcal{K}| + 3.$$

Denote by $\mathcal{P}$ the *projection* of $\mathcal{K}$ onto the plane $(y=0)$ parallel to the vector $w = \max(K_0)e_3 - e_2$. Note that $\mathcal{P}$ is a three dimensional set that lies on ℓ_0, ℓ_1 and ℓ_2, $\mathcal{P} \cap \ell_2 = \{e_2\}$ and $\mathcal{P} \setminus \{e_2\} \subseteq (y=0)$. We shall use the following notation. For every $0 \leq i \leq 2$ choose a set $P_i \subset \mathbb{Z}$ such that $\mathcal{P} \cap \ell_i = \{e_i + ze_3 \mid z \in P_i\}$ and denote the length of P_i by $l(P_i) = \max(P_i) - \min(P_i)$. It is clear that

$$l(P_0) = l(K_0) + l(K_2), \quad l(P_1) = l(K_1) \ \text{ and } \ l(P_2) = 0.$$

We claim that the 2 dimensional set $\mathcal{P}^* = \mathcal{P} \setminus \{e_2\}$ satisfies $|2\mathcal{P}^*| < 4|\mathcal{P}^*| - 6$. Indeed, in view of (1.16.1), [3, pp. 27], we have $|2\mathcal{K}| \geq |2\mathcal{P}|$ and $|\mathcal{K}| = |\mathcal{P}|$; it follows that the set $\mathcal{P}^*$ satisfies

$$|\mathcal{P}^*| = |\mathcal{P}| - 1 = |\mathcal{K}| - 1, \quad |2\mathcal{P}| = |2\mathcal{P}^*| + |\mathcal{P}^*| + 1 = |2\mathcal{P}^*| + |\mathcal{K}|$$

and therefore

$$|2\mathcal{P}^*| = |2\mathcal{P}| - |\mathcal{K}| \leq |2\mathcal{K}| - |\mathcal{K}| < 4|\mathcal{K}| - 10 = 4|\mathcal{P}| - 10 = 4|\mathcal{P}^*| - 6.$$

Assertion (iii) implies that the greatest common divisor of $P_0 \cup P_1$ is also equal to one. We may use now Theorem B from [8] and obtain $l(P_0) + l(P_1) \leq |2\mathcal{P}^*| - 2|\mathcal{P}^*| + 1$. Using $l(P_0) + l(P_1) = l(K_0) + l(K_1) + l(K_2)$, $|2\mathcal{P}^*| \leq |2\mathcal{K}| - |\mathcal{K}|$ and $|\mathcal{P}^*| = |\mathcal{P}| - 1 = |\mathcal{K}| - 1$ we get

$$l(K_0) + l(K_1) + l(K_2) \leq (|2\mathcal{K}| - |\mathcal{K}|) - 2|\mathcal{P}^*| + 1 = |2\mathcal{K}| - 3|\mathcal{K}| + 3.$$

This completes the proof of Theorem 1.1(b). ∎

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